Review II
Problem 3

Problem 3. Gauss Law(8)
A.Gauss Law in Native marie "GLM")

It considers the fallowing gueegtion:
Gungider a closed surface $S$ in $\mathbb{R}^{3}$ and a point I which in not an S.
$Q$. Ie $P$ insides
or is $P$ outstare $S$ ?
[S can be cempeicasted, the simplest case for us will be a sphere. I
Gauss provided a computatitinal way to decide this. For this he defined a Vector field $F_{p}$ that depends on the point? The value at a perish is

Remark. Notice that ere way the paint $P$ is special for thus vel et tor field is that FP is
defined ie meaningful every share an $\mathbb{R}^{3}$ excopet at the point $P$
TAFCR)P the decouninetern is O! I
TI. (GUM) $\underbrace{\text { SS }} F_{p} \cdot d \vec{S}= \begin{cases}\frac{4 \pi}{0} & \text { if } \frac{\text { Pis inside } S}{} \text {, } \\ \text { if } P \text { is outside }\end{cases}$ when $S$ is anieuded entwourds.
Our goal is to prove this Cains, ie. to explain why this is true.

Proof. It is based on the GDT.

$$
\begin{aligned}
& \text { sss } \operatorname{siv}(F) d v= \text { ss } F \cdot \overrightarrow{d S} . \\
& \text { w } \\
& \text { ow } \\
& \text { selig. } \text { (arientabibu outwards) }
\end{aligned}
$$

Wautscheck $\begin{aligned} & \text { SS Fp.d } \\ & \text { S elesed } \\ & \frac{S}{\text { compute }}\end{aligned}=$
(a) Chaese a ceordinde syspen in $\mathbb{R}^{3}$

Ys $Q$ sa thet $P$ is at the

$$
P=(0,0, a) \text { arigin! }
$$

$$
F_{P}(Q)=\frac{\overrightarrow{P Q}}{|\overrightarrow{P Q}|^{3}}=\frac{r}{|r|^{3}}
$$

Poeition vectar fer $Q$ is $v=\overrightarrow{P Q}$.
(b) $P$ onsside $S$ : $\quad \frac{S S}{S} F_{p}-d \vec{S}=$
$P$
$\theta$

$$
\begin{aligned}
& \text { did } \\
& \operatorname{div}\left(F_{P}\right)=\operatorname{div}\left(\frac{m}{\mid w)^{3}}\right)=(Q) \\
& P_{x}{ }^{\prime \prime}+Q_{y}+R_{z}>\frac{r}{1 v 13}=\frac{\langle x, y, z\rangle}{|r| 3}=\langle P, Q, R\rangle
\end{aligned}
$$

$$
\begin{gathered}
X_{(y)} z \text { coordindes } \\
\text { of } Q
\end{gathered}
$$

So:

$$
\begin{aligned}
P & =\frac{x}{151^{3}}= \\
& =\frac{x}{\sqrt{x^{2}+y^{2}+z^{2}}}
\end{aligned}
$$

so we need a ealid W whose bowndary is avr swrface $S$.

$$
\begin{aligned}
& \text { Cheese } W=\text { inside ofs } \\
& =\left\{\begin{array}{l}
\text { ropid encessed } \\
\text { by } S
\end{array} .\right. \\
& \text { SS } F_{p} \cdot d \vec{S}=S S F_{p} \cdot d \vec{S} \\
& =\operatorname{SSS}_{W} \text { divp dV } \\
& =835 \text { odv=0 }
\end{aligned}
$$

as promised by GLIT.

Dived computabion!

$$
\begin{aligned}
& \text { SS } F_{p}-\underbrace{d \vec{S}}_{n d \vec{S}}=\operatorname{SS}_{\substack{\text { splere of } \\
\text { vadies }}}^{W D^{3}} \cdot \frac{r}{w i} d A= \\
& =S \int \frac{1}{\log 1^{3}} \frac{\ln 1^{2}}{\ln 1} d A
\end{aligned}
$$

$$
\begin{aligned}
& =S \rho \frac{1}{\left.S\right|^{2}} d A=S S \frac{1}{R^{2}} d A=\frac{1}{R^{2}} \underbrace{\int \rho^{1} d \Delta}_{\text {area }} \\
& \mid r)=B \\
& =\frac{1 \overrightarrow{P Q} \mid}{R^{2}} \cdot \frac{4 \pi R^{2}}{}=4 \pi \quad C K_{\theta} \rho
\end{aligned}
$$

Cerrpute the area!
No thearems bat somple shape!
(d) Case when, $P$ is inside $S$


$$
\begin{aligned}
& S S F_{p} \cdot \overrightarrow{d s} \\
& S \\
& 11(x) \\
& \delta S F_{p}-d \vec{s}
\end{aligned}
$$

chaese $\varepsilon$ suall enough so that the ball $B_{\varepsilon}(P)$ is still inside $S$.
Lat $\tau=$ boundaris of $\theta g(P)$,
Than: $\quad$ is $\stackrel{S S}{T}$
reason if $w$ is the selid

$\int F_{P}$ is desined an $W$ :
$\left\{\begin{array}{l}F_{p} \text { is defined an problem at } \\ F_{p} \text { enly has probin wil }\end{array}\right.$
Then

$$
3 s_{w}^{3 g}\left(E_{p}\right) 2 \overrightarrow{3}=S F_{p}-2 s
$$

So:

$$
\begin{aligned}
& 0=s s_{w} 0 \mathrm{ds} \\
& \text { 6Dt aw } \\
& \text { appries } \\
& 0=\frac{9}{a w}=\frac{s}{3}-\frac{3}{1}, 30 \text { : } \\
& 2 W=+S^{2 W}-T \text { orievbafions: }
\end{aligned}
$$



Gaves Law in Phyistce (GLD)
The flux (ie. the total flow) of an elecritic field $F$ through a closed surfaces, equal the:
6T. total electric change inside is.

- Explanation.
there, it is know that a partide at a point $P$ that carries charge $C$ careses the electric farce I in space

$$
\begin{aligned}
& \text { such that } E(Q)=C \frac{\overrightarrow{P Q}}{\mid \overrightarrow{P Q} B} \\
& \text { Remark. } \frac{\overrightarrow{P Q}}{\sqrt{\overrightarrow{P Q} B}}=\frac{1}{|\overrightarrow{P Q}| 2} \cdot \frac{\overrightarrow{P Q}}{|\overrightarrow{P Q}|}
\end{aligned}
$$

$$
P_{Q} E(C)=\frac{1}{\text { distance }^{2}}=\frac{\text { unit vector }}{\text { in the died ias }}
$$ in the directions

from $p$ to,
This also appear in the formula for the gravitational fence.
[Eaten forces are proportions te one ever distance sgueared!J
Proof Consider a cleeeh surface 8 and charges $C_{i}$ at paints Pi for $t=l_{7}-i^{n}$.
 charge $C=c_{i}$.

$$
\begin{aligned}
& \text { Flux }=\int_{S}^{S S} d \vec{S}
\end{aligned}
$$

$$
\begin{aligned}
& =C \cdot \begin{array}{l}
\rho S \\
\rho
\end{array} F_{p}-d \vec{s}=C \begin{cases}4 \pi & p \text { insie } \\
0 & \text { acses }\end{cases}
\end{aligned}
$$

Case es ar poil.
So cousioer $P_{1, \ldots}, P_{n}, \ldots, C_{1}$

$$
\begin{aligned}
E & =C_{1} \cdot F_{p_{1}}+\cdots+C_{n} \cdot F_{P_{s}} \\
F \operatorname{Fen} & =S S E d S=\sum_{i=1}^{n} S S C_{i}-F_{p_{i}} d S \\
& =\sum_{i=1}^{n} C_{i} \cdot\left\{4 \pi p_{i}\right. \text { insides } \\
& =4 \pi \sum_{i} C_{i} \\
& =4 \pi \cdot \underbrace{P_{i}}_{\text {Total char as }} \\
& \text { insides So }
\end{aligned}
$$

